

# CHARACTERS, GENERICITY, AND MULTIPLICITY ONE FOR $U(3)$

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ABSTRACT. Let  $\psi : U \rightarrow \mathbb{C}^\times$  be a generic character of the unipotent radical  $U$  of a Borel subgroup of a quasisplit  $p$ -adic group  $G$ . The number (0 or 1) of  $\psi$ -Whittaker models on an admissible irreducible representation  $\pi$  of  $G$  was expressed by Rodier in terms of the limit of values of the trace of  $\pi$  at certain measures concentrated near the origin. An analogous statement holds in the twisted case. This twisted analogue is used in [F], p. 47, to provide a local proof of the multiplicity one theorem for  $U(3)$ . This asserts that each discrete spectrum automorphic representation of the quasisplit unitary group  $U(3)$  associated with a quadratic extension  $E/F$  of number fields occurs in the discrete spectrum with multiplicity one. It is pointed out in [F], p. 47, that a proof of the twisted analogue of Rodier's theorem does not appear in print. It is then given below. Detailing this proof is necessitated in particular by the fact that the attempt in [F], p. 48, at a global proof of the multiplicity one theorem for  $U(3)$ , although widely quoted, is incomplete, as we point out here.

Let  $E/F$  be a quadratic extension of  $p$ -adic fields,  $p \neq 2$ , and consider the basechange lifting from the quasisplit unitary group  $H = U(3, E/F)$  in 3 variables to  $G = GL(3, E)$ . Our main result is that there exists a family of suitably related functions  $\psi'_{H,n}$  on  $H = U(3, E/F)$  and  $\psi'_n$  on  $G = GL(3, E)$  which are supported near the origin, such that the traces  $\text{tr } \pi_H(\psi'_{H,n} dh)$  and twisted traces  $\text{tr } \pi(\psi'_n dg \times \sigma)$  stabilize for sufficiently large  $n$  and become equal to the Whittaker multiplicity (multiplicity in the space of Whittaker vectors) of the irreducible admissible representations  $\pi_H$  of  $H$  and  $\pi$  of  $G$ . This is a twisted analogue of a theorem of Rodier for the involution that defines  $H$  in  $G$ .

Our motivation for considering such a twisted analogue of Rodier's theorem is that we found a gap in an attempted global proof (of [F], p. 48) of multiplicity one theorem for automorphic representations of  $U(3, E/F)(\mathbb{A}_F)$ . We use our result to provide a local proof, as suggested in [F], p. 47.

This gap is different than the observation of Harder ([H], p. 173). The latter deals with the question of which representation in a packet is automorphic, assuming multiplicity one holds.

In Section 1 we state a theorem of Rodier and explain what are Whittaker models, characters, and the measures concentrated near the origin which enter the statement. We

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then state its twisted analogue of interest to us. In Section 2 we use these theorems to complete a key step in the proof of [F], p. 47, of multiplicity one theorem for the global quasisplit unitary group  $U(3)$ . In Section 3 we explain why the global proof of [F], p. 48, of this multiplicity one theorem is not complete. In Section 4 we recall the main lines of Rodier's proof. In Section 5 the twisted analogue is reduced to Rodier's theorem. This completes the proof of the theorems of Sections 1 and 2. In the Appendix, Section 6, we derive another description of the Whittaker multiplicity, in terms of the coefficient of the regular orbit in the germ expansion of the character.

## 1. WHITTAKER MODELS AND CHARACTERS

Rodier's theorem [R], p. 161, (for a split group  $H$ ) computes the number of  $\psi_H$ -Whittaker models of the admissible irreducible representation  $\pi_H$  of  $H$  in terms of values of the character  $\text{tr } \pi_H$  or  $\chi_{\pi_H}$  of  $\pi_H$  at the measures  $\psi_{H,n} dh$  which are supported near the origin.

We proceed to explain the notations to be used in Rodier's theorem. For simplicity and clarity, instead of working with a general connected reductive (quasi) split  $p$ -adic group  $H$  as in [R], we let  $H$  be a specific unitary group. To define it, we take  $G = \text{GL}(r, E)$ , where  $E/F$  is a quadratic extension of  $p$ -adic fields of characteristic zero,  $p \neq 2$ . Let  $x \mapsto \bar{x}$  denote the generator of  $\text{Gal}(E/F)$ . For  $g = (g_{ij})$  in  $G$  we put  $\bar{g} = (\bar{g}_{ij})$  and  ${}^t g = (g_{ji})$ . Then  $\sigma(g) = J^{-1} \bar{g}^{-1} J$ ,  $J = ((-1)^{i-1} \delta_{i, r+1-j})$ , defines an involution  $\sigma$  on  $G$ . The group  $H = G^\sigma$  of  $g \in G$  fixed by  $\sigma$  is a quasisplit unitary group  $U(r, E/F)$ .

Denote by  $\psi_H : U_H \rightarrow \mathbb{C}^\times$  a character on the unipotent upper triangular subgroup  $U_H$  of  $H$ . It is necessarily unitary, namely its values lie in the unit circle  $\mathbb{C}^1 = \{z \in \mathbb{C}; |z| = 1\}$ .

We assume that  $\psi_H$  is generic, namely nontrivial on each simple root subgroup. There is only one orbit of generic  $\psi_H$  under the action of the diagonal subgroup of  $H$  on  $U_H$  by conjugation. Hence we can and will work with the specific character  $\psi_H : U_H \rightarrow \mathbb{C}^1$ , defined by  $\psi_H((u_{ij})) = \psi(\sum_{1 \leq j < r} u_{j,j+1})$ . Here  $\psi : F \rightarrow \mathbb{C}^1$  is an additive character which is 1 on  $R$  and not identically 1 on  $\pi^{-1}R$ . Further,  $R$  is the ring of integers of  $F$ , and  $\pi$  is a generator of the maximal ideal of  $R$ . Note that  $u_{r-j, r-j+1} = \bar{u}_{j,j+1}$ .

By  $\psi_H$ -Whittaker vectors we mean vectors in the space of the induced representation  $\text{ind}_{U_H}^H(\psi_H)$ . They are the functions  $\varphi : H \rightarrow \mathbb{C}$  with  $\varphi(uhk) = \psi_H(u)\varphi(h)$ ,  $u \in U_H$ ,  $h \in H$ ,  $k \in K_\varphi$ , where  $K_\varphi$  is a compact open subgroup of  $H$  depending on  $\varphi$ , which are compactly supported on  $U_H \backslash H$ . The group  $H$  acts by right translation.

The multiplicity  $\dim_{\mathbb{C}} \text{Hom}_H(\text{ind}_{U_H}^H \psi_H, \pi_H)$  of any irreducible admissible representation  $\pi_H$  of  $H$  in the space of  $\psi_H$ -Whittaker vectors is known to be 0 or 1. In the latter case we say that  $\pi_H$  has a  $\psi_H$ -Whittaker model or that it is  $\psi_H$ -generic.

Let  $\mathcal{G}_0$  be the ring of  $r \times r$  matrices with entries in the ring of integers  $R_E$  of  $E$ . It is a subring of the ring  $\mathcal{G}$  of  $r \times r$  matrices with entries in  $E$ . Let  $d\sigma$  be the involution  $d\sigma(X) = -J^{-1} \bar{X} J$ . Its set of fixed points in  $\mathcal{G}$  is denoted by  $\mathcal{H}$ . Let  $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$  be the set of  $X$  in  $\mathcal{G}_0$  fixed by the involution  $d\sigma$ . Write  $H_n = \exp(\mathcal{H}_n)$ ,  $\mathcal{H}_n = \pi^n \mathcal{H}_0$ . For  $n \geq 1$  we have  $H_n = {}^t U_{H,n} A_{H,n} U_{H,n}$ , where  $U_{H,n} = U_H \cap H_n$ , and  $A_{H,n}$  is the group of diagonal matrices in  $H_n$ . Define a character  $\psi_{H,n} : H \rightarrow \mathbb{C}^1$ , supported on  $H_n$ , by  $\psi_{H,n}({}^t b u) = \psi(\sum_{1 \leq j < r} u_{j,j+1} \pi^{-2n})$ , at  ${}^t b \in {}^t U_{H,n} A_{H,n}$ ,  $u = (u_{ij}) \in U_{H,n}$ . Alternatively,

by

$$\psi_{H,n}(\exp X) = \text{ch}_{\mathcal{H}_n}(X) \psi(\text{tr}[X \pi^{-2n} \beta_H]),$$

where  $\text{ch}_{\mathcal{H}_n}$  indicates the characteristic function of  $\mathcal{H}_n = \pi^n \mathcal{H}_0$  in  $\mathcal{H}$ , and  $\beta_H$  is the  $r \times r$  matrix whose nonzero entries are 1 at the places  $(j, j-1)$ ,  $1 < j \leq r$ . Denote by  $e_{H_n}$  the constant measure of volume one supported on the compact subgroup  $H_n$  in the Hecke algebra of  $H$ , i.e.,  $e_{H_n} = |H_n|^{-1} \text{ch}_{H_n} dh$ , where  $|H_n|$  denotes the volume of  $H_n$  in  $dh$ .

Since  $\pi_H$  is admissible, for each test measure  $f_H dh$  ( $dh$  is a Haar measure on  $H$  and  $f_H$  is a locally constant compactly supported complex valued function on  $H$ ), the image of  $\pi_H(f_H dh)$  is finite dimensional and its trace  $\text{tr} \pi_H(f_H dh)$  is finite.

Rodier's theorem asserts

**Theorem 1.** *The multiplicity  $\dim_{\mathbb{C}} \text{Hom}_H(\text{ind}_{U_H}^H \psi_H, \pi_H)$  is equal to*

$$\lim_n \text{tr} \pi_H(\psi_{H,n} e_{H_n}).$$

In fact the limit stabilizes for sufficiently large  $n$ . Throughout this paper “ $= \lim_n a_n$ ” means “equals  $a_n$  for all sufficiently large  $n$ ”.

We need a twisted analogue of Rodier's theorem. It can be described as follows.

Let  $\pi$  be an admissible irreducible representation of  $G$  which is  $\sigma$ -invariant:  $\pi \simeq \sigma\pi$ , where  $\sigma\pi(\sigma(g)) = \pi(g)$ . Then there exists an intertwining operator  $A : \pi \rightarrow \sigma\pi$ , with  $A\pi(g) = \pi(\sigma(g))A$  for all  $g \in G$ . Since  $\pi$  is irreducible, by Schur's lemma  $A^2$  is a scalar which we may normalize by  $A^2 = 1$ . Thus  $A$  is unique up to a sign. Denote by  $G'$  the semidirect product  $G \rtimes \langle \sigma \rangle$ . Then  $\pi$  extends to  $G'$  by  $\pi(\sigma) = A$ .

Define  $\psi_E : E \rightarrow \mathbb{C}^1$  by  $\psi_E(x) = \psi(x + \bar{x})$ . Define a character  $\psi : U \rightarrow \mathbb{C}^1$  on the unipotent upper triangular subgroup  $U$  of  $G$  by  $\psi((u_{ij})) = \psi_E(\sum_{1 \leq j < r} u_{j,j+1})$ . This one dimensional representation has the property that  $\psi(\sigma(u)) = \psi(u)$  for all  $u$  in  $U$ . Note that  $\psi(u) = \psi_H(u^2)$  at  $u \in U_H = U \cap H$ . There is only one orbit of generic  $\sigma$ -invariant characters on  $U$  under the adjoint action of the group of  $\sigma$ -invariant diagonal elements in  $G$ .

Suppose that  $\pi$  is  $\psi$ -generic, namely  $\text{Hom}_{\mathbb{C}}(\text{ind}_U^G \psi, \pi) \neq \{0\}$ . Here  $\text{ind}_U^G \psi$  consists of the functions  $\varphi : G \rightarrow \mathbb{C}$  with  $\varphi(ug) = \psi(u)\varphi(g)$ ,  $u \in U$ ,  $g \in G$ , which are compactly supported on  $U \backslash G$ . Then we normalize  $A$  by  $A\varphi' = \sigma\varphi'$ , where  $\sigma\varphi(g) = \varphi(\sigma(g))$ , on the image  $\varphi'$  in  $\pi$  of the  $\varphi$ .

Write  $G_n = \exp(\mathcal{G}_n)$ , where  $\mathcal{G}_n = \pi^n \mathcal{G}_0$ . For  $n \geq 1$  we have  $G_n = {}^t U_n A_n U_n$ , where  $U_n = U \cap G_n$ , and  $A_n$  is the group of diagonal matrices in  $G_n$ . Define a character  $\psi_n : G \rightarrow \mathbb{C}^1$  supported on  $G_n$  by  $\psi_n({}^t b u) = \psi_E(\sum_{1 \leq j < r} u_{j,j+1} \pi^{-2n})$  where  ${}^t b \in {}^t U_n A_n$ ,  $u = (u_{ij}) \in U_n$ . Alternatively,  $\psi_n : G \rightarrow \mathbb{C}^1$  is defined by

$$\psi_n(\exp X) = \text{ch}_{\mathcal{G}_n}(X) \psi_E(\text{tr}[X \pi^{-2n} \beta])$$

where  $\beta$  is the  $r \times r$  matrix with entries 1 at the places  $(j, j-1)$ ,  $1 < j \leq r$ , and 0 elsewhere.

A first version of a  $\sigma$ -twisted analogue of Rodier's theorem asserts that the Whittaker multiplicity is equal to the twisted trace  $\text{tr} \pi(\psi_n e_{G_n} \times \sigma)$  for all sufficiently large  $n$ , where  $e_{G_n}$  is defined analogously to  $e_{H_n}$ .

A more useful version for us is stated in terms of the twisted character  $\chi_\pi^\sigma$ . Let us first restate Rodier's theorem this way.

Denote by  $\chi_{\pi_H}$  the *character* of  $\pi_H$ . It is a complex valued conjugacy invariant function on  $H$  which is locally constant on the regular set and locally integrable on  $H$  (Harish-Chandra [HC], Theorem 1) defined by  $\text{tr } \pi_H(f_H dh) = \int_H \chi_{\pi_H}(h) f_H(h) dh$  for all  $f_H dh$ .

Then Rodier's theorem can be stated as asserting that the Whittaker multiplicity of Theorem 1 is equal to

$$\lim_n \int_H \chi_{\pi_H}(h) \psi_{H,n}(h) e_{H_n}(h).$$

Analogously, the twisted character  $\chi_\pi^\sigma$  of  $\pi$  is a complex valued  $\sigma$ -conjugacy invariant function on  $G$  (that is, its value on  $\{hg\sigma(h)^{-1}\}$  is independent of  $h \in G$ ) which is locally constant on the  $\sigma$ -regular set ( $g$  with regular  $g\sigma(g)$ ), locally integrable (Clozel [C], Thm 1, p. 153) and defined by  $\text{tr } \pi(f dg) A = \int_G \chi_\pi^\sigma(g) f(g) dg$  for all test measures  $f dg$ .

The  $\sigma$ -twisted analogue of Rodier's theorem of interest to us is as follows. Let  $e_{G_n^\sigma}$  denote the constant measure of volume 1 supported on the compact subgroup  $G_n^\sigma = \{g = \sigma g; g \in G_n\}$  of  $G$ . As  $G_n^\sigma$  is  $H_n$ ,  $e_{G_n^\sigma}$  lies in the Hecke algebra of  $H$ .

**Theorem 2.** *The multiplicity  $\dim_{\mathbb{C}} \text{Hom}_{G'}(\text{ind}_U^G \psi, \pi) = \dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_U^G \psi, \pi)$  is equal to  $\int_{G_n^\sigma} \chi_\pi^\sigma(g) \psi_n(g) e_{G_n^\sigma}(g)$  for all sufficiently large  $n$ .*

*Remark.* Recall that  $\text{Hom}_G(\pi_1, \pi_2^\vee) = \text{Hom}_G(\pi_2, \pi_1^\vee)$  as both spaces can be identified with the space of  $(\pi_1, \pi_2)$ -invariant bilinear forms. The contragredient  $(\text{ind}_H^G \rho)^\vee$  is  $\text{Ind}_H^G(\frac{\Delta_G}{\Delta_H} \rho^\vee)$  ([BZ1], 2.25(c); Ind indicates non compact induction,  $H$  is a closed subgroup of  $G$ ,  $\rho$  is a representation of  $H$ ,  $\pi$  of  $G$ ). Frobenius reciprocity asserts

$$\text{Hom}_G(\text{ind}_H^G \rho, \pi) = \text{Hom}_H(\frac{\Delta_H}{\Delta_G} \rho, \pi|_H)$$

and equivalently  $\text{Hom}_G(\pi, \text{Ind}_H^G \rho) = \text{Hom}_H(\pi|_H, \rho)$  ([BZ1], (2.28) and (2.29)).

## 2. MULTIPLICITY ONE FOR $U(3)$

Let us recall how the Theorems are used in the proof of Proposition 3.5 of [F], p. 47. Thus in this Section we work only with  $H = U(r, E/F)$  and  $G = \text{GL}(r, E)$ ,  $r = 2$  or  $3$ , of Section 1. We are given a square integrable irreducible admissible representation  $\rho$  of the quasisplit group  $U(2, E/F)$ . Its stable basechange to  $\text{GL}(2, E)$  is denoted by  $\tau$ . The unstable basechange is  $\tau \otimes \kappa$ . Let  $\pi = I(\tau \otimes \kappa)$  be the normalizedly induced representation of  $\text{GL}(3, E)$ . This  $\pi$  is invariant under the involution  $\sigma$  (ours and of [F]). It is generic. For all matching measures  $f dg$  and  $f_H dh$  on  $G = \text{GL}(3, E)$  and  $H = U(3, E/F)$ , using an identity of trace formulae and orthogonality relations for characters [F] obtains an identity

$$\text{tr } \pi(f dg \times \sigma) = (2m + 1) \sum_{\pi_H} \text{tr } \pi_H(f_H dh).$$

The sum ranges over finitely many (in fact, two times the cardinality of the packet of  $\rho$ ) inequivalent square integrable irreducible admissible representations  $\pi_H$  of  $U(3, E/F)$ . The number  $m$  is a nonnegative integer, independent of  $\pi_H$ .

**Proposition 3.5 of [F].** *The nonnegative integer  $m$  is zero, and there is a unique generic  $\pi_H$  in the sum. The other  $2[\{\rho\}] - 1$  representations  $\pi_H$  are not generic.*

Note that our  $\pi, \pi_H, fdg, f_H dh, G, H$ , are denoted in [F] by  $\Pi, \pi, \phi dg', fdg, G', G$ .

*Proof.* The identity for all matching test measures  $fdg$  and  $f_H dh$  implies an identity of characters:

$$\chi_\pi^\sigma(\delta) = (2m + 1) \sum_{\pi_H} \chi_{\pi_H}(\gamma)$$

for all  $\delta \in G = GL(3, E)$  with regular norm  $\gamma \in H = U(3, E/F)$ . Note that  $\delta \mapsto \chi_\pi^\sigma(\delta)$  is a stable  $\sigma$ -conjugacy class function on  $G$ , while  $\gamma \mapsto \sum_{\pi_H} \chi_{\pi_H}(\gamma)$  is a stable conjugacy class function on  $H$ . We use Theorem 2 with  $G = GL(3, E)$  and  $H = G^\sigma$ . Then  $G_n^\sigma = H_n$ . On  $\delta \in G_n^\sigma$ , the norm  $N\delta$  of the stable  $\sigma$ -conjugacy class  $\delta$  is just the stable conjugacy class of  $\delta^2$ . Hence  $\chi_\pi^\sigma(\delta) = (2m + 1) \sum_{\pi_H} \chi_{\pi_H}(\delta^2)$  at  $\delta \in G_n^\sigma = H_n$ .

We claim that for  $\delta = \exp X$ ,  $X \in \mathcal{G}_n^\sigma = \mathcal{H}_n$ , we have

$$\psi_E(\text{tr}[X\pi^{-2n}\beta]) = \psi(\text{tr}[2X\pi^{-2n}\beta_H]).$$

For this we note that  $\beta = \beta_H$  and  $\psi_E(x) = \psi(x + \bar{x})$ .

Moreover we claim that  $\psi_n(\delta) = \psi_{H,n}(\delta^2)$  for  $\delta \in G_n^\sigma = H_n$ . For this we note that  $\psi_n(tbu) = \psi_E((x + y)\pi^{-2n})$  if  $u = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ , and this is  $= \psi(2(x + \bar{x})\pi^{-2n})$  if  $y = \bar{x}$ . But  $\psi_{H,n}(tbu) = \psi((x + \bar{x})\pi^{-2n})$  at such  $u \in U_H$  (thus with  $y = \bar{x}$ ).

Now  $d(g^2) = dg$  when  $p \neq 2$ . It follows that

$$\begin{aligned} 1 &= \dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_U^G \psi, \pi) = \int_{G_n^\sigma} \chi_\pi^\sigma(\delta) \psi_n(\delta) e_{G_n^\sigma}(\delta) \\ &= \int_{H_n} (2m + 1) \sum_{\pi_H} \chi_{\pi_H}(\delta^2) \psi_{H,n}(\delta^2) e_{H_n}(\delta^2) \\ &= (2m + 1) \sum_{\pi_H} \dim_{\mathbb{C}} \text{Hom}_H(\text{ind}_{U_H}^H \psi_H, \pi_H). \end{aligned}$$

Hence  $m = 0$  and there is just one generic  $\pi_H$  in the sum ( $\dim_{\mathbb{C}} \neq 0$ , necessarily  $= 1$ ).  $\square$

The excluded case of  $p = 2$  might follow on counting the factors of 2 in our argument.

We repeat the conclusion of [F]. Note that the unrestricted trace identity is proven in [F1], and the fundamental lemma in [F2]. Both proofs employ simple methods: the usage of regular-Iwahori functions in [F1] removes the need to compute and compare weighted orbital integrals, and explicit double coset decomposition reduces the fundamental lemma in [F2] to elementary volume computations.

**Corollary.** *Each discrete spectrum automorphic representation of  $U(3, E/F)(\mathbb{A})$  occurs in the discrete spectrum with multiplicity one. Each packet (defined in [F]) of such an infinite dimensional representation contains precisely one irreducible representation which is generic. Each packet (defined in [F]) of tempered admissible representations of  $U(3, E/F)$  contains precisely one irreducible representation which is generic. There are no generic representations in any (nontempered) quasipacket, locally and globally.*

This type of argument, relating the number of Whittaker models to the values of the characters at suitable measures supported near the origin, was first employed in [FK], which appeared in 1987, in an analogous situation of the metaplectic correspondence between  $GL(r)$  and its  $n$ -fold covering group.

*Remark.* On the space  $L^2(GL(n, F) \backslash GL(n, \mathbb{A}))$  of automorphic forms  $\phi$ , we define the involution  $\sigma$  by  $(r(\sigma)\phi)(g) = \phi(\sigma(g))$ . On the space of Whittaker functions  $W$  ( $W(ng) = \psi(n)W(g)$ ,  $g \in GL(n, \mathbb{A})$ ,  $n \in N(\mathbb{A})$ , where  $N$  denotes the unipotent upper triangular subgroup of  $GL(n)$ ), we choose the natural action of  $\sigma$ , by  ${}^\sigma W(g) = W(\sigma g)$ . The map  $\phi \mapsto W_\phi$ , where  $W_\phi(g) = \int_{N(\mathbb{A}) \backslash N(\mathbb{A})} \phi(ng) \overline{\psi}(n) dn$ , respects the action of  $\sigma$ . Thus the global normalization of the action of  $\sigma$  is the product of the local normalizations  ${}^\sigma W_v(g) = W_v(\sigma g)$ .

Note also that underlying the character identity is an embedding of the local representation  $\pi$ , which is  $\sigma$ -elliptic, as a component of a global  $\sigma$ -invariant cuspidal representation  $\Pi$  of  $GL(3, \mathbb{A}_E)$  where  $F$  is a totally imaginary field. The character identity follows from using the trace formulae identity, applying “generalized linear independence of characters” and thus isolating  $\Pi$ , and further,  $\pi$ . The normalization of  $\sigma$  on the generic ( $\Pi$  and)  $\pi$  implies that no sign occurs in the character identity.

Had we not fixed the choice of the sign of  $r(\sigma)$  at all places, the twisted character in our trace identity (at our place) might in principle be replaced by its negative. However, the identity

$$-\mathrm{tr} \pi(fdg \times \sigma) = (2m+1) \sum_{\pi_H} \mathrm{tr} \pi_H(f_H dh), \quad -\chi_\pi^\sigma(\delta) = (2m+1) \sum_{\pi_H} \chi_{\pi_H}(\gamma),$$

cannot hold, as evaluating it with our  $fdg = \psi_n e_{G_n^\sigma}$  and  $f_H dh = \psi_{H,n} e_{H_n}$ ,  $n$  large, would give  $-1$  on the left, and a nonnegative integer on the right. This provides an independent verification that our normalization of the sign of  $r(\sigma)$  is correct.

### 3. INCOMPLETE GLOBAL PROOF

The second proof of Proposition 3.5 of [F], on p. 48, is global, but incomplete. The false assertion is on lines 21-22: “Proposition 8.5(iii) (p. 172) and 2.4(i) of [GP] imply that for some  $\pi$  with  $m(\pi) \neq 0$  above, we have  $m(\pi) = 1$ ”. Indeed, [GP], Prop. 2.4, defines  $L_{0,1}^2$  to be the orthocomplement in the space  $L_0^2$  (of cusp forms) of “all hypercusp forms”, and claims: “(i)  $L_{0,1}^2$  has multiplicity 1”. ([GP], 8.5 (iii), asserts that  $\pi$  is in  $L_{0,1}^2$ ). Now the sentence of [F], p. 48, l. 21-22 assumes that [GP], 2.4(i), means that any irreducible  $\pi$  in  $L_{0,1}^2$  occurs in  $L_0^2$  with multiplicity one. But the standard techniques of [GP], 2.4, show only that any irreducible  $\pi$  in  $L_{0,1}^2$  occurs in  $L_{0,1}^2$  with multiplicity one. *A-priori* there can exist

$\pi'$  in  $L_0^2$ , isomorphic and orthogonal to  $\pi \subset L_{0,1}^2$ . In such a case we would have  $m(\pi) > 1$ . Such a  $\pi'$  is locally generic (all of its local components are generic), isomorphic to a generic cuspidal  $\pi$ , and the question boils down to whether this implies that  $\pi'$  is generic (the linear form  $L(\phi) = \int_{U_H(F) \backslash U_H(\mathbb{A})} \phi(u) \psi_H(u) du$  is nonzero on  $\pi \subset L_0^2$ ). This last claim might follow on using the theory of the Theta correspondence, but this has not been done as yet. In summary, a clear form of [GP], 2.4(i) is: “Any irreducible  $\pi$  in  $L_{0,1}^2$  occurs in  $L_{0,1}^2$  with multiplicity one.” In the analogous situation of  $\mathrm{GSp}(2)$  such a statement is made in [So]. It is not sufficiently strong to be useful for us.

We noticed that the global argument of [F], p. 48, which was first proposed in a preprint version of [F] in 1983, is incomplete while generalizing it in [F3] to the context of the symplectic group, where work of Kudla, Rallis, Langlands, Shahidi on the Siegel-Weil formula and on L-functions is available to show that a locally generic cuspidal representation which is equivalent at almost all places to a generic cuspidal representation is generic. A local proof, based on a twisted analogue of Rodier’s result, is also used in [F4], in the context of the symmetric square lifting.

#### 4. REVIEW OF RODIER’S PROOF

We shall reduce Theorem 2 to Theorem 1 for  $G$  (not  $H$ ), so we begin by recalling the main lines in Rodier’s proof in the context of  $G$ . Choose  $d = \mathrm{diag}(\pi^{-r+1}, \pi^{-r+3}, \dots, \pi^{r-1})$ . It lies in the unitary group, namely  $\sigma(d) = d$ , since  $\pi$  is in  $F$ . Put  $V_n = d^n G_n d^{-n}$  and  $\psi_n(v) = \psi_n(d^{-n} v d^n)$  ( $v \in V_n$ ). Recall that  $\psi_n$  is defined to be supported on  $G_n$ . Note that  $\sigma(G_n) = G_n$ ,  $\sigma(U_n) = U_n$ ,  $\sigma\psi_n = \psi_n$ , and that the entries in the  $j$ th line ( $j \neq 0$ ) above or below the diagonal of  $v = (v_{ij})$  in  $V_n$  lie in  $\pi^{(1-2j)n} R_E$  (thus  $v_{i,i+j} \in \pi^{(1-2j)n} R_E$  if  $j > 0$ , and also when  $j < 0$ ). Thus  $V_n \cap U$  is a  $\sigma$ -invariant strictly increasing sequence of compact and open subgroups of  $U$  whose union is  $U$ , while  $V_n \cap ({}^t U H)$  – where  ${}^t U H$  is the lower triangular subgroup of  $G$  – is a strictly decreasing sequence of compact open subgroups of  $G$  whose intersection is the element  $I$  of  $G$ . Note that  $\psi_n = \psi$  on  $V_n \cap U$ .

Consider the induced representations  $\mathrm{ind}_{V_n}^G \psi_n$ , and the intertwining operators

$$A_n^m : \mathrm{ind}_{V_n}^G \psi_n \rightarrow \mathrm{ind}_{V_m}^G \psi_m,$$

$$(A_n^m \varphi)(g) = ((e_{V_m} \psi_m) * \varphi)(g) = \int_G \psi_m(u) \varphi(u^{-1}g) e_{V_m}(u)$$

( $g$  in  $G$ ,  $\varphi$  in  $\mathrm{ind}_{V_n}^G \psi_n$ ,  $e_{V_m} = |V_m|^{-1} 1_{V_m} dg$ ,  $|V_m|$  denotes the volume of  $V_m$ ,  $1_{V_m}$  denotes the characteristic function of  $V_m$ ). For  $m \geq n \geq 1$  we have

$$(A_n^m \varphi)(g) = ((e_{V_m \cap U} \psi) * \varphi)(g) = \int_G \psi(u) \varphi(u^{-1}g) e_{V_m \cap U}(u).$$

Hence  $A_m^\ell \circ A_n^m = A_n^\ell$  for  $\ell \geq m \geq n \geq 1$ . So  $(\mathrm{ind}_{V_n}^G \psi_n, A_n^m \ (m \geq n \geq 1))$  is an inductive system of representations of  $G$ . Denote by  $(I, A_n : \mathrm{ind}_{V_n}^G \psi_n \rightarrow I) \ (n \geq 1)$  its limit.

The intertwining operators  $\phi_n : \text{ind}_{V_n}^G \psi_n \rightarrow \text{ind}_U^G \psi$ ,

$$(\phi_n(\varphi))(g) = (\psi 1_U * \varphi)(g) = \int_U \psi(u) \varphi(u^{-1}g) du,$$

satisfy  $\phi_m \circ A_n^m = \phi_n$  if  $m \geq n \geq 1$ . Hence there exists a unique intertwining operator  $\phi : I \rightarrow \text{ind}_U^G \psi$  with  $\phi \circ A_n = \phi_n$  for all  $n \geq 1$ . Proposition 3 of [R] asserts that

**Lemma 1.** *The map  $\phi$  is an isomorphism of  $G$ -modules.*  $\square$

**Lemma 2.** *There exists  $n_0 \geq 1$  such that  $\psi_n * \psi_m * \psi_n = |V_n| |V_m \cap V_n| \psi_n$  for all  $m \geq n \geq n_0$ .*

*Proof.* This is Lemma 5 of [R]. We review its proof (the first displayed formula in the proof of this Lemma 5, [R], p. 159, line -8, should be erased).

There are finitely many representatives  $u_i$  in  $U \cap V_m$  for the cosets of  $V_m$  modulo  $V_n \cap V_m$ . Denote by  $\varepsilon(g)$  the Dirac measure in a point  $g$  of  $G$ . Consider

$$(\varepsilon(u_i) * \psi_n 1_{V_m \cap V_n})(g) = \int_G \varepsilon(u_i)(gh^{-1})(\psi_n 1_{V_m \cap V_n})(h) dh = \psi_n(u_i^{-1}g) = \psi_m(u_i)^{-1} \psi_m(g).$$

Note here that if the left side is nonzero, then  $g \in u_i(V_m \cap V_n) \subset V_m$ . Conversely, if  $g \in V_m$ , then  $g \in u_i(V_m \cap V_n)$  for some  $i$ . Hence  $\psi_m = \sum_i \psi_m(u_i) \varepsilon(u_i) * \psi_n 1_{V_m \cap V_n}$ , thus

$$\psi_n * \psi_m * \psi_n = \sum_i \psi_m(u_i) \psi_n * \varepsilon(u_i) * \psi_n 1_{V_m \cap V_n} * \psi_n.$$

Since  $\psi_n 1_{V_m \cap V_n} * \psi_n = |V_m \cap V_n| \psi_n$ , this is

$$= \sum_i \psi_m(u_i) |V_m \cap V_n| \psi_n * \varepsilon(u_i) * \psi_n.$$

But the key Lemma 4 of [R] asserts that  $\psi_n * \varepsilon(u) * \psi_n \neq 0$  implies that  $u \in V_n$ . Hence the last sum reduces to a single term, with  $u_i = 1$ , and we obtain

$$= |V_m \cap V_n| \psi_n * \psi_n = |V_m \cap V_n| |V_n| \psi_n.$$

This completes the proof of the lemma.  $\square$

**Lemma 3.** *For an inductive system  $\{I_n\}$  we have  $\text{Hom}_G(\varinjlim I_n, \pi) = \varprojlim \text{Hom}_G(I_n, \pi)$ .*

*Proof.* See, e.g., Rotman [Ro], Theorem 2.27. It is also verified in [R].  $\square$

**Corollary.** *We have  $\dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_U^G \psi, \pi) = \lim_n |G_n|^{-1} \text{tr } \pi(\psi_n dg)$ .*

*Proof.* As the  $\dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_{V_n}^G \psi_n, \pi)$  are increasing with  $n$ , if they are bounded we get that they are independent of  $n$  for sufficiently large  $n$ . Hence the left side of the corollary is equal to  $\lim_n \dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_{V_n}^G \psi_n, \pi)$ . This is equal to  $\lim_n \dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_{G_n}^G \psi_n, \pi)$  since  $\psi_n(v) = \psi_n(d^{-n} v d^n)$ . This is equal to  $\lim_n \dim_{\mathbb{C}} \text{Hom}_{G_n}(\psi_n, \pi|_{G_n})$  by Frobenius reciprocity. This is equal to the right side of the corollary since  $|G_n|^{-1} \pi(\psi_n dg)$  is a projection from  $\pi$  to the space of  $\xi$  in  $\pi$  with  $\pi(g)\xi = \psi_n(g)\xi$  ( $g \in G_n$ ), a space whose dimension is then  $|G_n|^{-1} \text{tr } \pi(\psi_n dg)$ .  $\square$



## 5. THE TWISTED CASE

We now reduce Theorem 2 to Theorem 1 for  $G$ . Note that since  $\sigma\psi_n = \psi_n$ , the representations  $\text{ind}_{V_n}^G \psi_n$  are  $\sigma$ -invariant, where  $\sigma$  acts on  $\varphi \in \text{ind}_{V_n}^G \psi_n$  by  $\varphi \mapsto \sigma\varphi$ ,  $(\sigma\varphi)(g) = \varphi(\sigma g)$ . Similarly  $\sigma\psi = \psi$  and  $\text{ind}_U^G \psi$  is  $\sigma$ -invariant. We then extend these representations  $\text{ind}$  of  $G$  to the semidirect product  $G' = G \rtimes \langle \sigma \rangle$  by putting  $(i(\sigma)\varphi)(g) = \varphi(\sigma(g))$ .

Let  $\pi$  be an irreducible admissible representation of  $G$  which is  $\sigma$ -invariant. Thus there exists an intertwining operator  $A : \pi \rightarrow \sigma\pi$ , where  $\sigma\pi(g) = \pi(\sigma(g))$ , with  $A\pi(g) = \pi(\sigma(g))A$ . Then  $A^2$  commutes with every  $\pi(g)$  ( $g \in G$ ), hence  $A^2$  is a scalar by Schur's lemma, and can be normalized to be 1. This determines  $A$  up to a sign. We extend  $\pi$  from  $G$  to  $G' = G \rtimes \langle \sigma \rangle$  by putting  $\pi(\sigma) = A$  once  $A$  is chosen.

If  $\text{Hom}_G(\text{ind}_U^G \psi, \pi) \neq 0$ , its dimension is 1. Choose a generator  $\ell : \text{ind}_U^G \psi \rightarrow \pi$ . Define  $A : \pi \rightarrow \pi$  by  $A\ell(\varphi) = \ell(i(\sigma)\varphi)$ . Then

$$\text{Hom}_G(\text{ind}_U^G \psi, \pi) = \text{Hom}_{G'}(\text{ind}_U^G \psi, \pi).$$

Similarly we have  $\text{Hom}_G(\text{ind}_{V_n}^G \psi_n, \pi) = \text{Hom}_{G'}(\text{ind}_{V_n}^G \psi_n, \pi)$ .

The right side in the last equality can be expressed as

$$\text{Hom}_{G'}(\text{ind}_{G_n}^G \psi_n, \pi) = \text{Hom}_{G'_n}(\psi'_n, \pi|_{G'_n}) \quad (G'_n = G_n \rtimes \langle \sigma \rangle).$$

The last equality follows from Frobenius reciprocity, where we extended  $\psi_n$  to a character  $\psi'_n$  on  $G'_n$  by  $\psi'_n(\sigma) = 1$ . Thus  $\psi'_n = \psi_n^1 + \psi_n^\sigma$ , with  $\psi_n^\alpha(g \times \beta) = \delta_{\alpha\beta}\psi_n(g)$ ,  $\alpha, \beta \in \{1, \sigma\}$ .

In this case  $\text{Hom}_{G'_n}(\psi'_n, \pi|_{G'_n})$  is isomorphic to the space  $\pi_1$  of vectors  $\xi$  in  $\pi$  with  $\pi(g)\xi = \psi_n(g)\xi$  for all  $g$  in  $G'_n$ . In particular  $\pi(g)\xi = \psi_n(g)\xi$  for all  $g$  in  $G_n$ , and  $\pi(\sigma)\xi = \xi$ . Clearly  $|G'_n|^{-1}\pi(\psi'_n dg')$  is a projection from the space of  $\pi$  to  $\pi_1$  (it is independent of the choice of the measure  $dg'$ ). Its trace is then the dimension of the space  $\text{Hom}$ . We conclude a twisted analogue of the theorem of [R]:

**Proposition 1.** *The integer  $\dim_{\mathbb{C}} \text{Hom}_{G'}(\text{ind}_U^G \psi, \pi)$  is equal to  $|G'_n|^{-1} \text{tr} \pi(\psi'_n dg')$  for all sufficiently large  $n$ .  $\square$*

Note that  $G'_n$  is the semidirect product of  $G_n$  and the two-element group  $\langle \sigma \rangle$ . With the natural measure assigning 1 to each element of the discrete group  $\langle \sigma \rangle$ , we have  $|G'_n| = 2|G_n|$ . The result is then, for all sufficiently large  $n$ ,

$$\frac{1}{2} \text{tr} \pi(\psi_n e_{G_n}) + \frac{1}{2} \text{tr} \pi(\psi_n e_{G_n} \times \sigma)$$

(as  $\psi'_n = \psi_n^1 + \psi_n^\sigma$ ,  $\psi_n^1 = \psi_n$  and  $\text{tr} \pi(\psi_n^\sigma dg) = \text{tr} \pi(\psi_n dg \times \sigma)$ ). By (the nontwisted) Rodier's Theorem 1,

$$\dim_{\mathbb{C}} \text{Hom}_G(\text{ind}_U^G \psi, \pi) = \lim_n \text{tr} \pi(\psi_n e_{G_n}),$$

we conclude

**Proposition 2.** *We have  $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \psi, \pi) = \lim_n \operatorname{tr} \pi(\psi_n e_{G_n} \times \sigma)$  for all  $\sigma$ -invariant irreducible representations  $\pi$  of  $G$ .*  $\square$

The terms in the limit on the right can be written in terms of Harish-Chandra's twisted character, as

$$\int_G \chi_{\pi}^{\sigma}(g) \psi_n(g) e_{G_n}(g).$$

Again, put  $e_{G_n^{\sigma}} = |G_n^{\sigma}|^{-1} \operatorname{ch}_{G_n^{\sigma}} dg$  where  $\operatorname{ch}_{G_n^{\sigma}}$  is the characteristic function of  $G_n^{\sigma}$  in  $G$ .

**Proposition 3.** *The last displayed integral is equal to*

$$\int_{G_n^{\sigma}} \chi_{\pi}^{\sigma}(g) \psi_n(g) e_{G_n^{\sigma}}(g).$$

*Proof.* Consider the map  $G_n^{\sigma} \times G_n^{\sigma} \backslash G_n \rightarrow G_n$ ,  $(u, k) \mapsto k^{-1} u \sigma(k)$ . It is a closed immersion. More generally, given a semisimple element  $s$  in a group  $G$ , we can consider the map  $Z_{G^0}(s) \times Z_{G^0}(s) \backslash G^0 \rightarrow G^0$  by  $(u, k) \mapsto k^{-1} u s k s^{-1}$ . Our example is:  $(s, G) = (\sigma, G_n \times \langle \sigma \rangle)$ .

Our map is in fact an analytic isomorphism since  $G_n$  is a small neighborhood of the origin, where the exponential  $e : \mathcal{G}_n \rightarrow G_n$  is an isomorphism. Indeed, we can transport the situation to the Lie algebra  $\mathcal{G}_n$ . Thus we write  $k = e^Y$ ,  $u = e^X$ ,  $\sigma(k) = e^{(d\sigma)(Y)}$ ,  $k^{-1} u \sigma(k) = e^{X-Y+(d\sigma)(Y)}$ , up to smaller terms. Here  $(d\sigma)(Y) = -J^{-1} t \bar{Y} J$ . So we just need to show that  $(X, Y) \mapsto X - Y + (d\sigma)(Y)$ ,  $Z_{\mathcal{G}_n}(\sigma) + \mathcal{G}_n(\operatorname{mod} Z_{\mathcal{G}_n}(\sigma)) \rightarrow \mathcal{G}_n$ , is bijective. But this is obvious since the kernel of  $(1 - d\sigma)$  on  $\mathcal{G}_n$  is precisely  $Z_{\mathcal{G}_n}(\sigma) = \{X \in \mathcal{G}_n; (d\sigma)(X) = X\}$ .

Changing variables on the terms on the right of Proposition 2 we get the equality:

$$\begin{aligned} & \int_{G_n} \chi_{\pi}^{\sigma}(g) \psi_n(g) e_{G_n}(g) \\ &= |G_n|^{-1} \int_{G_n^{\sigma}} \int_{G_n^{\sigma} \backslash G_n} \chi_{\pi}^{\sigma}(k^{-1} u \sigma(k)) \psi_n(k^{-1} u \sigma(k)) dk du. \end{aligned}$$

But  $\sigma \psi_n = \psi_n$ ,  $\psi_n$  is a homomorphism (on  $G_n$ ),  $G_n$  is compact, and  $\chi_{\pi}^{\sigma}$  is a  $\sigma$ -conjugacy class function, so we end up with the expression of the proposition. Note that  $\chi_{\pi}^{\sigma}$  is locally integrable on  $G_n^{\sigma}$  and locally constant on its regular set by the character relation stated in the proof of Prop. 3.5 of [F] above. The proposition, and Theorem 2, follow.  $\square$

## 6. APPENDIX. GERMS OF TWISTED CHARACTERS

Harish-Chandra [HC] showed that  $\chi_{\pi}$  is locally integrable (Thm 1, p. 1) and has a germ expansion near each semisimple element  $\gamma$  (Thm 5, p. 3), of the form:

$$\chi_{\pi}(\gamma \exp X) = \sum_{\mathcal{O}} c_{\gamma}(\mathcal{O}, \pi) \hat{\mu}_{\mathcal{O}}(X).$$

Here  $\mathcal{O}$  ranges over the nilpotent orbits in the Lie algebra  $\mathcal{M}$  of the centralizer  $M$  of  $\gamma$  in  $G$ ,  $\mu_{\mathcal{O}}$  is an invariant distribution supported on the orbit  $\mathcal{O}$ ,  $\hat{\mu}_{\mathcal{O}}$  is its Fourier transform with

respect to a symmetric nondegenerate  $G$ -invariant bilinear form  $B$  on  $\mathcal{M}$  and a selfdual measure, and  $c_\gamma(\mathcal{O}, \pi)$  are complex numbers. Both  $\mu_{\mathcal{O}}$  and  $c_\gamma(\mathcal{O}, \pi)$  depend on a choice of a Haar measure  $d_{\mathcal{O}}$  on the centralizer  $Z_G(X_0)$  of  $X_0 \in \mathcal{O}$ , but their product does not. The  $X$  ranges over a small neighborhood of the origin in  $\mathcal{M}$ . We shall be interested only in the case of  $\gamma = 1$ , and thus omit  $\gamma$  from the notations. The size of the domain where the germ expansion holds is studied in Waldspurger [W].

Suppose that  $G$  is quasisplit over  $F$ , and  $U$  is the unipotent radical of a Borel subgroup  $B$ . Let  $\psi : U \rightarrow \mathbb{C}^\times$  be the nondegenerate character of  $U$  (its restriction to each simple root subgroup is nontrivial) specified in Rodier [R], p. 153. The number  $\dim_{\mathbb{C}} \text{Hom}(\text{ind}_U^G \psi, \pi)$  of  $\psi$ -Whittaker functionals on  $\pi$  is known to be zero or one. Let  $\mathcal{G}_0$  be a selfdual lattice in the Lie algebra  $\mathcal{G}$  of  $G$ . Denote by  $\text{ch}_0$  the characteristic function of  $\mathcal{G}_0$  in  $\mathcal{G}$ . Rodier [R], p. 163, showed that there is a regular nilpotent orbit  $\mathcal{O} = \mathcal{O}_\psi$  such that  $c(\mathcal{O}, \pi)$  is not zero iff  $\dim_{\mathbb{C}} \text{Hom}(\text{ind}_U^G \psi, \pi)$  is one, in fact  $\widehat{\mu}_{\mathcal{O}}(\text{ch}_0)c(\mathcal{O}, \pi)$  is one in this case. Alternatively put, normalizing  $\mu_{\mathcal{O}}$  by  $\widehat{\mu}_{\mathcal{O}}(\text{ch}_0) = 1$ , we have  $c(\mathcal{O}, \pi) = \dim_{\mathbb{C}} \text{Hom}(\text{ind}_U^G \psi, \pi)$ . This is shown in [R] for all  $p$  if  $G = \text{GL}(r, F)$ , and for general quasisplit  $G$  for all  $p \geq 1 + 2 \sum_{\alpha \in S} n_\alpha$ , if the longest root is  $\sum_{\alpha \in S} n_\alpha \alpha$  in a basis  $S$  of the root system. A generalization of Rodier's theorem to degenerate Whittaker models and nonregular nilpotent orbits is given in Mœglin-Waldspurger [MW]. See [MW], I.8, for the normalization of measures. In particular they show that  $c(\mathcal{O}, \pi) > 0$  for the nilpotent orbits  $\mathcal{O}$  of maximal dimension with  $c(\mathcal{O}, \pi) \neq 0$ . For applications to minimal representations see Savin [S].

Harish-Chandra's results extend to the twisted case. The twisted character is locally integrable (Clozel [C], Thm 1, p. 153), and there exist unique complex numbers  $c^\theta(\mathcal{O}, \pi)$  ([C], Thm 3, p. 154) with  $\chi_\pi^\theta(\exp X) = \sum_{\mathcal{O}} c^\theta(\mathcal{O}, \pi) \widehat{\mu}_{\mathcal{O}}(X)$ . Here  $\mathcal{O}$  ranges over the nilpotent orbits in the Lie algebra  $\mathcal{G}^\theta$  of the group  $G^\theta$  of the  $g \in G$  with  $g = \theta(g)$ . Further,  $\mu_{\mathcal{O}}$  is an invariant distribution supported on the orbit  $\mathcal{O}$  (it is unique up to a constant, not unique as stated in [HC], Thm 5, and [C], Thm 3);  $\widehat{\mu}_{\mathcal{O}}$  is its Fourier transform, and  $X$  ranges over a small neighborhood of the origin in  $\mathcal{G}^\theta$ .

In this section we compute the expression displayed in Proposition 3 using the germ expansion  $\chi_\pi^\sigma(\exp X) = \sum_{\mathcal{O}} c^\sigma(\mathcal{O}, \pi) \widehat{\mu}_{\mathcal{O}}(X)$ . This expansion means that for any test measure  $f d\mathbf{g}$  supported on a small enough neighborhood of the identity in  $G$  we have

$$\int_{\mathcal{G}^\sigma} f(\exp X) \chi_\pi^\sigma(\exp X) dX = \sum_{\mathcal{O}} c^\sigma(\mathcal{O}, \pi) \int_{\mathcal{O}} \left[ \int_{\mathcal{G}^\sigma} f(\exp X) \psi(\text{tr}(XZ)) dX \right] d\mu_{\mathcal{O}}(Z).$$

Here  $\mathcal{O}$  ranges over the nilpotent orbits in  $\mathcal{G}^\sigma$ ,  $\mu_{\mathcal{O}}$  is an invariant distribution supported on the orbit  $\mathcal{O}$ ,  $\widehat{\mu}_{\mathcal{O}}$  is its Fourier transform. The  $X$  range over a small neighborhood of the origin in  $\mathcal{G}^\sigma$ . Since we are interested only in the case of the unitary group, and to simplify the exposition, we take  $G = \text{GL}(r, E)$  and the involution  $\sigma$  whose group of fixed points is the unitary group  $H = \text{U}(r, E/F)$ . In this case there is a unique regular nilpotent orbit  $\mathcal{O}_0$ .

We normalize the measure  $\mu_{\mathcal{O}_0}$  on the orbit  $\mathcal{O}_0$  of  $\beta$  in  $\mathcal{G}^\sigma$  by the requirement that  $\widehat{\mu}_{\mathcal{O}_0}(\text{ch}_0^\sigma)$  is 1, thus that  $\int_{\beta + \pi^n \mathcal{G}_0^\sigma} d\mu_{\mathcal{O}_0}(X) = q^{n \dim(\mathcal{O}_0)}$  for large  $n$ . Equivalently a measure on an orbit  $\mathcal{O} \simeq G/Z_G(Y)$  ( $Y \in \mathcal{O}$ ) is defined by a measure on its tangent space  $m = \mathcal{G}/Z_{\mathcal{G}}(Y)$  ([MW], p. 430) at  $Y$ , taken to be the selfdual measure with respect to the symmetric bilinear nondegenerate  $F$ -valued form  $B_Y(X, Z) = \text{tr}(Y[X, Z])$  on  $m$ .

**Proposition 4.** *If  $\pi$  is a  $\sigma$ -invariant admissible irreducible representation of  $G$  and  $\mathcal{O}_0$  is the regular nilpotent orbit in  $\mathcal{G}^\sigma$ , then the coefficient  $c^\sigma(\mathcal{O}_0, \pi)$  in the germ expansion of the  $\sigma$ -twisted character  $\chi_\pi^\sigma$  of  $\pi$  is equal to*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \psi, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_U^G \psi, \pi).$$

*This number is one if  $\pi$  is generic, and zero otherwise.*

*Proof.* We compute the expression displayed in Proposition 3 as in [MW], I.12. It is a sum over the nilpotent orbits  $\mathcal{O}$  in  $\mathcal{G}^\sigma$ , of  $c^\sigma(\mathcal{O}, \pi)$  times

$$|G_n^\sigma|^{-1} \widehat{\mu}_{\mathcal{O}}(\psi_n \circ e) = |G_n^\sigma|^{-1} \mu_{\mathcal{O}}(\widehat{\psi_n \circ e}) = |G_n^\sigma|^{-1} \int_{\mathcal{O}} \widehat{\psi_n \circ e}(X) d\mu_{\mathcal{O}}(X).$$

The Fourier transform (with respect to the character  $\psi_E$ ) of  $\psi_n \circ e$ ,

$$\widehat{\psi_n \circ e}(Y) = \int_{\mathcal{G}^\sigma} \psi_n(\exp Z) \overline{\psi}_E(\operatorname{tr} ZY) dZ = \int_{\mathcal{G}_n^\sigma} \psi_E(\operatorname{tr} Z(\pi^{-2n}\beta - Y)) dZ,$$

is the characteristic function of  $\pi^{-2n}\beta + \pi^{-n}\mathcal{G}_0^\sigma = \pi^{-2n}(\beta + \pi^n\mathcal{G}_0^\sigma)$  multiplied by the volume  $|\mathcal{G}_n^\sigma| = |G_n^\sigma|$  of  $\mathcal{G}_n^\sigma$ . Hence we get

$$= \int_{\mathcal{O} \cap (\pi^{-2n}(\beta + \pi^n\mathcal{G}_0^\sigma))} d\mu_{\mathcal{O}}(X) = q^{n \dim(\mathcal{O})} \int_{\mathcal{O} \cap (\beta + \pi^n\mathcal{G}_0^\sigma)} d\mu_{\mathcal{O}}(X).$$

The last equality follows from the homogeneity result of [HC], Lemma 3.2, p. 18. For sufficiently large  $n$  we have that  $\beta + \pi^n\mathcal{G}_0^\sigma$  is contained only in the orbit  $\mathcal{O}_0$  of  $\beta$ . Then only the term indexed by  $\mathcal{O}_0$  remains in the sum over  $\mathcal{O}$ , and

$$\int_{\mathcal{O}_0 \cap (\beta + \pi^n\mathcal{G}_0^\sigma)} d\mu_{\mathcal{O}_0}(X) = \int_{\beta + \pi^n\mathcal{G}_0^\sigma} d\mu_{\mathcal{O}_0}(X)$$

equals  $q^{-n \dim(\mathcal{O}_0)}$  (cf. [MW], end of proof of Lemme I.12). The proposition follows.  $\square$

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